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Evolution Semigroups and Hamiltonian Flows

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The operatorial calculus of Feinsilver is extended to a class of Hamiltonians possessing terms depending on the position variables.

1. INTRODUCTION

In [4] Feinsilver presented an operator calculus that allows one, among other things, to compute transition semigroups in terms of naturally associated Hamiltonian flows, but in his work the infinitesimal generators of the transition semigroups, or their associated Hamiltonians, depend only on the "momentum variables." Here we extend his techniques to situations in which the Hamiltonian may depend on the position variables. This will allow us to treat a larger class of processes and to bring into play the theory of canonical transformations.

This paper is to be the first of a series, very much modelled on [4], and we shall assume familiarity with its notations, result and techniques. Below we state the basic assumptions to be kept in mind throughout. In Section 2 we extend some of the results in [4] to out situation and we give a classical model for the Girsanov transformation: it corresponds to a canonical transformation. After this we study how transition functions "change under canonical transformations."

In Section 3 we work out some examples, after which in Section 4 we do a brief study of moment systems.

Let $H(x, p)$, the Hamiltonian function, be a real analytic function defined on $\mathbb{R}^{2n}$ such that $H(x, p) = T(p) + a(x) \cdot p + V(x)$ or reducible to such form by means of canonical transformations (see [7] for a crash course in classical mechanics or [1] for full details). We shall assume that the canonical equations

$$\dot{x}_i \equiv dx_i/dt = \partial H/\partial p_i, \qquad \dot{p}_i \equiv dp_i/dt = -\partial H/\partial x_i \tag{1.1}$$

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have a global solution through each initial point (x, p), which defines a flow $\Phi_t(x, p)$ on $\mathbb{R}^{2n}$. By $x(t)$ or $\Phi_t^1(x, p)$ we will denote the first n components of $\Phi_t(x, p)$.

A function $F(x, P, t)$ will be called the generating function of the canonical transformation $(x, p) \to (Q, P)$, if the equations

$$Q_i = \partial F/\partial P_i, \qquad p_i = \partial F/\partial x_i \tag{1.2}$$

can be solved for (Q, P) in terms of (x, p) and back. In this case the new Hamiltonian function and canonical equations are

$$\tilde{H} = H + \partial F/\partial t \tag{1.3a}$$

$$\dot{Q}_i = \partial\tilde{H}/\partial p_i, \qquad \dot{p}_i = -\partial\tilde{H}/\partial Q_i. \tag{1.3b}$$

Associated to $H(x, p)$ we put $G = H(x, D)$, with $D = -(\partial/\partial x_1, ..., \partial/\partial x_n)$, and assume the existence of a "transition function" $p_t(x, y)$ such that

$$P_t f(x) \equiv (e^{tG} f)(x) \equiv \int p_t(x, y) f(y)\, dy \tag{1.4}$$

defines a semigroup, on an appropriately large class of functions f, having infinitesimal generator G, i.e., $\partial P_t/\partial t = GP_t = P_t G$. To finish, we assume the existence of a positive function Ω_0, the vacuum function, such that $P_t\Omega_0 = \Omega_0$ (or $G\Omega_0 = 0$).

2. Operator Calculus

Here we extend some of the basic operator calculus of [4] to our setup. Basic to the subject is the idea of thinking of functions as multiplication operators acting on the vacuum function. Feinsilver does it by taking $\Omega_0 = 1$, but one can also reobtain $f(x)$ as $(f\Omega_0)(x)/\Omega_0(x)$. Let us begin with an extension of the

GLL (Generalized Leibnitz Lemma). *Let $K(x, D)$ denote an operator such that $K(x, p)$ is analytic and the D's are to the right. Then*

$$K(x, D) \circ U(x) = \sum_{(m)} \frac{\partial^{(m)} U}{\partial x^{(m)}} \frac{\partial^{(m)} K}{\partial p^{(m)}}(x, D)/(m)!$$

where we are using the multi-index notation, i.e.,

$$\frac{\partial^{(n)}}{\partial (x)^n} = \frac{\partial^{m_1}}{\partial x^{m_1}} \cdots \frac{\partial^{m_n}}{\partial x_n^{m_n}}, \qquad (m)! = m_1! \cdots m_n!, \quad \textit{etc.},$$

$$\textit{for} \quad (m) = (m_1, ..., m_n).$$

Proof. As in [4], it suffices to consider, with $e_k(x) = \exp ik \cdot x$,

$$(K(x, D) \circ e_k)\, e_{k'}(x) = K(x, D)\, e_{k+k}(x)$$

$$= K(x, ik + ik')\, e_{k+k'}(x) = \sum_{(m)} \frac{(ik)^{(m)}}{(m)} e^{ik \cdot x} \frac{\partial^{(m)}}{\partial p^{(m)}} K(x, ik')\, e^{ik' \cdot x}$$

$$= \sum_{(m)} \frac{\partial^{(m)}}{\partial x^{(m)}} e^{ik \cdot x} \frac{\partial^{(m)}}{\partial p^{(m)}} K(x, D)\, e^{ik'}/(m)!. \quad \blacksquare$$

A computation that will be needed repeatedly below is contained in

PROPOSITION 2.1. *Let $l(p)$ denote an analytic, real valued function on $\mathbb{R}^n$ and $F(p) = \nabla l(p)$. Then for $\alpha, k \in \mathbb{R}^n$, and $a, b \in \mathbb{R}$, $b \neq 0$,*

$$\exp(a\alpha \cdot F(D) + b\alpha \cdot x)\, e_k = e^{b\alpha \cdot x} \exp \frac{a}{b} \{l(ik + b\alpha) - l(ik)\}\, e^{ik \cdot x} \quad (2.2)$$

Proof. To obtain this result we use the following version of Trotter's product formula [9],

$$\exp \alpha \cdot (aF(D) + bx)\, e_k = \lim_{n \to \infty} \left(\exp \frac{a\alpha \cdot F(D)}{n} \exp \frac{b\alpha \cdot x}{n}\right)^n e_k$$

and observe that

$$\exp \frac{a\alpha \cdot F(D)}{n} \cdot \exp x \cdot \left(ik + \frac{b\alpha}{n}\right) = \exp \frac{a\alpha}{n} \cdot F\left(ik + \frac{b\alpha}{n}\right) \exp x \cdot \left(ik + \frac{b\alpha}{n}\right).$$

Now, iterating one obtains that the left-hand side of (2.2) equals

$$\lim_{n \to 0} \exp \sum_{j=1}^{n} \frac{a\alpha}{n} \cdot F\left(ik + \frac{b}{n} i\right) \exp(b\alpha + ik) \cdot x$$

which, after converting the Riemann sum into an integral, yields (2.2). $\blacksquare$

Another variation on the same theme, with a similar proof, is

PROPOSITION 2.3. *Let $h(x)$ be a smooth function and let $K(x) = \nabla h(x)$. Then for $x, k \in \mathbb{R}^n$, $b, \alpha \neq 0$ in $\mathbb{R}$,*

$$\exp \alpha \cdot (aD + bK(x))\, e_k = \exp ia\alpha \cdot k \exp \frac{b}{a} \{h(x + a\alpha) - h(x)\}\, e^{ik \cdot x}. \quad (2.4)$$

An important special case of this corresponding to $h(x) = \frac{1}{2}x^2$ is where (2.4) becomes

$$\exp \alpha \cdot (aD + bx)\, e_k = \exp i\alpha a \cdot k \exp b\alpha \cdot x + \tfrac{1}{2} ba\alpha^2 \exp ik \cdot x. \quad (2.5)$$

These two results are an extension of Proposition 8 in [4]. The next result is the basic tool for the examples in the next section. It is an analogue of the duality of the Heisenberg and Schroedinger representation in quantum mechanics [6], and it extends the comments preceding Proposition 8 in [4].

Starting from $p_t(x, y)$ put $q_t(x, y) = \Omega_0^{-1}(x)\, p_t(x, y)\, \Omega_0(y)$ and define the semigroup Q_t by

$$Q_t f(x) = \int q_t(x, y)\, f(y)\, dy = \Omega_0(x)^{-1}\, P_t f\Omega_0(x) = \Omega_0(x)^{-1}(e^{tG} f\Omega_0)(x). \tag{2.6}$$

Certainly, when $q_t(x, y)$ is known, $p_t(x, y)$ can be obtained from

$$p_t(x, y) = \Omega_0(x)\, q_t(x, y)\, \Omega_0^{-1}(y).$$

PROPOSITION 2.7. *With the above notations, and for H of one of the two forms $H = T(p) + ax \cdot p + V(x)$ or $H = \frac{1}{2}p^2 + a(x)\cdot p + V(x)$, where $a \in \mathbb{R}$ or $a(x) = \nabla A$, $A(x)$ being real valued analytic,*

$$(Q_t f)(x) = \Omega_0(x)^{-1}(f(C^+)\, \Omega_0)(x) \tag{2.8}$$

where $C^+ \equiv C^+(t)$ is $\Phi_t^1(x_1 p)$ with p replaced by D. (The reason for the notation will become apparent in Section 4.)

Proof (i) In the style of [4], $\Omega_0(x)^{-1}(e^{tH} f\Omega_0)(x) = \Omega_0(x)^{-1}(e^{tH} f e^{-tH} e^{tH}\Omega_0)(x) = \Omega_0(x)^{-1}(f(C^+)\, \Omega 0)(x)$.

Proof (ii) We shall verify that both sides have the same derivative with respect to t. We do it for $H = T(p) + ax\cdot p + V(x)$, the other case having a similar proof. Put $U_t(x) = Q_t f(x)$, then it suffices to verify that

$$\Omega_0(x)^{-1}\, G(U_t\Omega_0)(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{\Omega_0(x)^{-1}(U_t(C^+(\varepsilon))\, \Omega_0)(x) - U_t(x)\}.$$

Now, up to $O(\varepsilon^2)$, $C^+(\varepsilon) = x + \varepsilon\nabla_p H\,|_{p=D} = (1+\varepsilon a)\, x + \varepsilon\nabla T(p)\,|_{p=D} = (1+\varepsilon a)\, x + \varepsilon l(D)$. Then

$$\lim \frac{1}{\varepsilon}\{\Omega_0(x)^{-1}\, U_t(C^+(\varepsilon)\, \Omega_0)(x) - U_t(x)\}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{\Omega_0(x)^{-1}\} \iint \hat{U}_t(k)\, \hat{\Omega}_0(k')\{e^{ik\cdot C^+(\varepsilon)} e^{ik!x} - 1\}\, dk\, dk'$$

where dk denotes the n-dimensional volume element in $\mathbb{R}^n$, $\hat{U}_t(k) = \int U_t(x)\, C^{-ikx}\, dx/(2\pi)^n$ and the same for $\hat{\Omega}_0(k)$.

Now, from (2.2) and then from GLL it will follow that, after taking limit as $\varepsilon \to 0$, the last expression yields

$$\Omega_0(x)^{-1} \iint \hat{U}_t(k)\, \hat{\Omega}_0(k')\{H(x, ik-ik') - H(x, ik)\}\, e^{ikx} e^{ik'x}\, dk\, dk'$$

$$= \Omega_0(x)^{-1} \sum_{(m)\geqslant 1} \int \frac{(ik)^{(m)}}{(m)!} \hat{U}_t(k)\, e^{ik\circ x}\, dk$$

$$\times \frac{\partial^{(m)}}{\partial p^{(m)}} H(x, -ik')\, e^{ik'\cdot x} \hat{\Omega}_0(k')\, dk'$$

$$= \Omega_0(x)^{-1} \sum_{(m)\geqslant 1} \frac{\partial^{(m)}}{\partial x^{(m)}} U_t(x) \frac{\partial^{(m)}}{\partial p^{(m)}} H(x, D)\, \Omega_0(x)$$

$$= \Omega_0(x)^{-1} \left(\sum_{(m)\geqslant 1} \frac{\partial^{(m)}}{\partial x^{(m)}} U_t(x) \frac{\partial^{(m)}}{\partial p^{(m)}} H(x, D)\, \Omega_0 \right)(x)$$

$$= \Omega_0(x)^{-1}(H(x, D)\, U_t \Omega_0)(x) \equiv \Omega_0(x)^{-1}\, G(U_t \Omega_0)(x)$$

where $(m) \geqslant 1$ denotes the multi-index in which at least one element is $\geqslant 1$, and the second step next to the last follows from our assumption on Ω_0, *i.e.*, $H^{(0)}(x, D)\, \Omega_0 = G\Omega_0 = 0$.

Let us now examine the transformation $p_t(x, y) \to q_t(x, y) = \Omega_0(x)^{-1}\, p_t(x, y)\, \Omega_0(y)$, $H = \frac{1}{2}p^2 + a(x)\cdot p + V(x)$ and p_t is the transition semigroup of a Markov process X_t, then Q_t is the transition semigroup of the process obtained from X_t by subordination with respect to $\Omega_0(X_t)/\Omega_0(X_0)$, see Dynkin [3], and the infinitesimal generator of Q_t is

$$\tilde{G}f(x) = \Omega_0(x)^{-1}(Gf\Omega_0)(x) = \tfrac{1}{2}\Delta f + h(x)\cdot \nabla f \qquad (2.9)$$

where Δ denotes the Laplace operator and $h(x) = \nabla \ln \Omega_0(x) - a(x)$.

The relationship between that and subordination with respect to $m_t = m_t^1 m_t^2$, with $m_t^2 = \exp - \int_0^t V(X_s)\, ds$ and

$$m_t^1 = \exp \int_0^t \left(\frac{\nabla \Omega_0}{\Omega_0}\right)(X_s)\, dX_s - \frac{1}{2}\int_0^t \left(\frac{\nabla \Omega}{\Omega_0}\right)^2 (X_s)\, ds,$$

has already been noticed, see [2] for a review. Here $dX_s = sB_t - a(X_t)\, dt$, B_t being the standard brownian motion on $\mathbb{R}^n$. Of course $P_t f(x) = E^x\{f(X_t) \exp \int_0^t V(X_s)\, ds\}$ and the connection with the first subordination comes from noticing that due to $H(x, D)\, \Omega_0 = 0$,

$$\Omega_0(X_t)/\Omega_0(X_0) = m_t = m_t^1 m_t^2.$$

The connection with classical mechanics is the following: the canonical transformation $(x, p) \to (Q, P) = (x, p - \nabla \ln \Omega_0)$ is generated by $F(x, P) = x \cdot P - \nabla \ln \Omega_0(x)$ and is such that the new Hamiltonian is $\tilde{H}(Q, P) = \frac{1}{2}(P - \nabla \ln \Omega_0)^2 + a(x) \cdot (P - \nabla \ln \Omega_0) + V(x)$ which after replacing P_i by $-\partial/\partial Q_i$ becomes $\tilde{G}$ as given by (2.9). To finish this disgression we note that even when the subordination with respect to multiplicative functionals cannot be applied, for example, when $H = T(p) + a(x) \cdot p + V$ for general $T(p)$, the infinitesimal generator $\tilde{G}$ can still be obtained from G by means of the canonical transformation $(x, p) \to (Q, P) = (x, p - \nabla \ln \Omega_0)$. This is the content of

LEMMA 2.10. *Let $h(x) = \nabla \ln \Omega_0(x)$. Then for any multi-index (m)*

$$(D + h(x))^{(m)} f(x) = \Omega_0(x)^{-1}((D)^{(m)} f \Omega_0)(x). \tag{2.11}$$

Proof. When $|m| = \sum m_i = 1$ it is obvious. Assume (2.11) is true for some (m), then if $\delta_i = (0,..., 1,..., 0)$, $i = 1, 2,..., n$,

$$\begin{aligned} \Omega_0(x)^{-1}((D)^{(m+\delta_i)} f \Omega_0)(x) &= \Omega_0(x)^{-1}(D)^{(m)} \Omega_0(D^{\delta_i} f + h(x)) \\ &= (D + h)^{(m)}(D^{\delta_i} f + h(x)) = (D + h)^{(m+\delta_i)} f. \quad \blacksquare \end{aligned}$$

This degression about canonical transformations can be cast into a framework analogous to that of quantum mechanics (see [8], for example). To motivate, notice that if $F(x, P) = \sum_{i=1}^n \phi_i(x) P_i$ with $\phi = (\phi_1,..., \phi_n)$ bein a diffeomorphism of $\mathbb{R}$ onto itself (or onto some appropriate open subset of $\mathbb{R}^n$) and if $\tilde{f}: \mathbb{R}^n \to \mathbb{R}$, then if $\hat{\tilde{f}}(k) = \int e^{ikQ} \tilde{f}(Q)\, dQ$,

$$f(x) = \tilde{f}(\phi(x)) = \int \frac{e^{\bar{F}(x, ik)}}{(2\pi)^n} \hat{\tilde{f}}(k)\, dk, \qquad \bar{F}(x, ik) = -F(x, ik), \tag{2.12}$$

and the nice thing about he substitution transform (2.12) is that it is invertible, i.e.,

LEMMA 2.13. *With the same notation as above, if $F'(Q, p) = \sum_1^n \phi_i^{-1}(Q) P_i$ then*

$$\tilde{f}(Q) = \int \frac{e^{\bar{F}'(Q, ik)}}{(2\pi)} \hat{f}(k)\, dk. \tag{2.14}$$

Proof. It suffices to notice that

$$\int \frac{e^{\bar{F}'(Q, ik)}}{(2\pi)^n} e^{ikx}\, dk = \delta(\phi^{-1}(Q) - x). \quad \blacksquare$$

Comment. The role of the above canonical transformations will become clearer in example (e) below.

This whole setup can be extended to canonical transformations of the type

$$F(x, P_1 t) = \sum_1^n \phi_i(x, t) P_i - \psi(x, t) \tag{2.15}$$

where $\phi(, t)$: $\mathbb{R}^n \to \mathbb{R}^n$ has a differentiable universe, smooth in t, etc., then

$$F'(Q, P, t) = \sum \phi_i^{-1}(Q, t) p_i + \psi(\phi^{-1}(Q, t), t) \tag{2.16}$$

generates the canonical transformation inverse to (2.15), and Lemma 2.13 becomes

LEMMA 2.17. *With the same notations as above, if*

$$f(x, t) = \int \frac{e^{\bar{F}(x, ik, t)}}{(2\pi)^n} \hat{\tilde{f}}(k, t)\, dk = e^{\psi(x,t)} \tilde{f}(\phi(x, t)) \tag{2.18}$$

then

$$\tilde{f}(Q, t) = \int \frac{e^{\bar{F}'(Q, ik, t)}}{(2\pi)^n} \hat{f}(k, t)\, dk. \tag{2.19}$$

Proof. Similar to that of Lemma 2.13. ∎

Comments. Of course Lemmas 2.13 and 2.17 can be proved by trivial substitution. The whole point of (2.18) and (2.19) is to have a scheme allowing for the transformations themselves and for possible extension to more general transformations. This is carried out below. We obtain in this way a theory of representations of a subgroup of the group of canonical transformations.

Let us examine now how to relate solutions of $\partial\tilde{\rho}/\gamma t = \tilde{G}\tilde{\rho}$ to solutions of $\partial\rho/\partial t = G\rho$, where the Hamiltonians $\tilde{H}$ and H, associated to $\tilde{G}$ and G, respectively, are related by (1.3a), i.e., $\tilde{H} = H + \partial F/\partial t$. This is the content of

PROPOSITION 2.20. *Assume that* $F(x, P)$ *is given by* (2.15) *and that*

$$\begin{aligned}\tilde{G} &= \tilde{H}(Q, \partial/Q)\\ &= H(\phi^{-1}(Q), (\nabla F)(\phi^{-1}(Q), \partial/\partial Q)) + (\partial F/\partial t)(\phi^{-1}(Q), \partial/\partial Q, t)\\ &= \tilde{T}(\partial/\partial Q) + \tilde{U}(Q)\end{aligned}$$

and let $\tilde{\rho}(Q, t)$ *satisfy* $\partial\tilde{\rho}/\partial t = \tilde{G}\rho(Q, t)$, *then*

$$\rho(x, t) = e^{\psi(x,t)}\tilde{\rho}(\phi(x, t), t) = \int e^{\bar{F}(x,ik,t)}\hat{\rho}(k, t)\, dk/(2\pi)^n$$

satisfies $\partial\rho/\partial t = G\rho(x, t)$.

Proof. By taking Fourier transforms of $\partial\tilde{\rho}/\partial t = \tilde{G}\tilde{\rho}$ note that

$$\partial\hat{\rho}(k, t)/\partial t = (\tilde{T}(-ik) + \tilde{U}(-i\nabla_k))\,\hat{\rho}(k, t)$$

where the symbols have an obvious meaning. Now,

$$\begin{aligned}
\partial\rho/\partial t &= \int e^{\bar{F}(x,ik,t)}\left\{\frac{\partial\bar{F}}{\partial t}(x, ik, t) + \tilde{T}(-ik) + \tilde{U}(-i\nabla_k)\right\}\hat{\rho}(k, t)\, dk/(2\pi)^n \\
&= \int e^{\bar{F}(x,ik,t)}\left\{-\frac{\partial F}{\partial t}\left(x, \frac{\partial}{\partial Q}, t\right) + \tilde{T}\left(\frac{\partial}{\partial Q}\right) + \tilde{U}(Q)\right\} \\
&\quad \times \int e^{ik\cdot Q}\tilde{\rho}(Q, t) dQ\, dk/(2\pi)^n \\
&= \int \frac{e^{\bar{F}(x,ik,t)}}{(2\pi)^n}\{\quad\}\int e^{iQ\cdot k}\tilde{\rho}(Q, t)\, dQ\, dk
\end{aligned}$$

where $\{\quad\}$ stands for

$$\begin{aligned}
&\left\{H\left(\phi^{-1}(Q), (\nabla F)\left(\phi^{-1}(Q), \frac{\partial}{\partial Q}, t\right)\right) + \left(\frac{\partial F}{\partial t}\right)\left(\phi^{-1}(Q), \frac{\partial}{\partial Q}, t\right)\right. \\
&\quad \left. - \frac{\partial F}{\partial t}\left(x, \frac{\partial}{\partial Q}, t\right)\right\}.
\end{aligned}$$

Now, using the representation

$$\int e^{\bar{F}(x,ik)}e^{ikQ}\, dk/(2\pi)^n = e^{\psi(x,t)}\delta(\phi(x) - Q)$$

and integrating with respect to Q we obtain the desired result since the last two terms cancel out. ▌

To close this circle of ideas, and to tie up with Lemma 2.10 and the comments preceding it, note that when $F(x, D) = x\cdot P - \ln\Omega_0(x)$, $Q_t\tilde{f} = e^{t\tilde{G}}\tilde{f}$ can be computed with the aid of $P_t = e^{tG}$ as follows: apply P_t to $f(x) = e^{\ln\Omega_0}\tilde{f}(x) = \Omega_0(x)\tilde{f}(x)$ and express in terms of Q $(=x)$ again $Q_t\tilde{f} = e^{-\ln\Omega_0(x)}(P_t\Omega_0 f)(x) = \Omega_0(x)^{-1}(P_t\Omega_0 f(x)$. But the most important application is contained in

THEOREM 2.21. *Assume $H = H(p)$, i.e., the classical system is integrable. Then $S(x, p, t) = x \cdot P - H(P)\,t$ is the canonical transformation "bringing the system to rest" and $\rho(x, t) = \int \exp - S(x, ik, t)\hat{f}(k)\,dk/(2\pi)^n$ satisfies*

$$\frac{\partial \rho}{\partial t} = H(D)\,\rho, \qquad D = -(\partial/\partial x_1, ..., \partial/\partial x_n), \tag{2.22}$$

and $p(x, t) \to f(x)$ as $t \to 0$.

Proof.

$$\frac{\partial p}{\partial t} = \int \left(-\frac{\partial S}{\partial t}\right) \exp - S(x, k, t)\hat{f}(k)\,dk/(2\pi)^n$$

$$= \int H(ik)\{\exp - ikx \exp + H(ik)\,t\}\,\hat{f}(k)\,dk/(2\pi)^n = H(D)\,\rho(x, t)$$

where we used $H(ik) = -(\partial S/\partial t)(x, ik, t)$. The limit as $t \to 0$ is obvious. ∎

Comments. The open problem now is, given a system described by $H(x, p)$ and a transformation $(x, p) \to (Q, P)$, generated, say, by $F(x, P)$, what is the $\tilde{f}(Q)$ corresponding to $f(x)$? When the x's and p's are not mixed, the results are contained in Lemma 2.17 above, and to treat the general case note the following

LEMMA 2.23. *Let $F^{(1)}(x, p^1)$, $F^{(2)}(x^1, p^2)$ be the generating functions for the transformations $(x, p) \to (x^1, p^1) \to (x^2, p^2)$, then $F(x, p^2) = F^{(1)}(x, p') - x^1 \cdot p^1 + F^{(2)}(x^1, p^2)$ generates the composite transformation $(x, p) \to (x^2, p^2)$.*

Proof. Just use (1.2). ∎

Comment. When $F^{(2)}(x^1, p^2)$ is the inverse of $F^{(1)}(x, p')$ then $F(x, P^2) = x \cdot P^2$ generates the identity transformation. ∎

LEMMA 2.24. *Let $F(x, P, t) = x \cdot \phi(P, t) + g(P, t)$ be such that for every t $\phi(\cdot, t)$: $R^n \to R^n$ has an inverse, smooth in x and t. Then $F'(Q, p, t) = q \cdot \phi^{-1}(p, t) - g(\phi^{-1}(p, t), t)$ generates the canonical transformation inverse to that generated by $F(x, P, t)$. Also, Lemma 2.17 is valid in this case.*

Proof. Similar to that of Lemma 2.17. ∎

LEMMA 2.25. *The canonical transformation generated by $F(x, P, t) = \phi(x, t) \cdot \psi(P, t) + h(x, t) + g(P, t)$ can be obtained by composing the transformations generated by $F^1(x, P, t) = P^1 \cdot \phi(x, t) + h(x, t)$ and $F^2(Q^1, P, t) = Q^1 \cdot \psi(P, t) + g(P, t)$, where for each t, $\phi(\cdot\, t)$ and $\psi(P, t)$, have inverses, smooth in both variables.*

Proof. Just apply (2.22). ∎

PROPOSITION 2.26. *For $F(x, P, t)$ as in Lemma 2.25, and $\tilde{f}(Q, t)$ being given, define*

$$f(x, t) = \int e^{\tilde{F}(x, ik, t)} \hat{\tilde{f}}(k, t)\, dk/(2\pi)^n.$$

Then

$$\tilde{f}(Q, t) = \int e^{F(Q, ik, t)} \hat{f}(k, t)\, dk/(1\pi)^n.$$

Proof. Apply Lemmas 2.25, 2.24 and 2.17 in appropriate order. ∎

Comment. Although Proposition 2.26 does not represent the most general case, i.e., that of arbitrary $F(x, P, t)$, it nevertheless covers a wide variety of cases.

3. EXAMPLES

(a) *The one-dimensional oscillator process.* Let $H(x, p)$ be $\frac{1}{2}(p^2 - \omega^2 x^2 + \omega)$. Now $\Omega_0(x) = \exp -(\omega x^2/2)$ is annihilated by $G = \frac{1}{2}(D^2 - \omega^2 x^2 + \omega)$. The canonical equations are easily integrable, yielding $x(t) = x \cosh \omega t + p/\omega \sinh \omega t$. Set $C^+ = bx + aD$ with $b = \cosh \omega t$, $a = \sinh \omega t/\omega$. Now

$$f(C^+)\, \Omega_0(x) = \iint \hat{f}(k)\, \hat{\Omega}_0(k')\, e^{ik \cdot C^+} e^{ik' \cdot x}\, dk\, dk'$$

and with the aid of (2.4) or (2.2) we obtain

$$f(C^+)\, \Omega_0(x) = \iint \hat{f}(k)\, e^{-k^2 \frac{\sinh \omega t \cosh \omega t}{2}} e^{ikx} \Omega_0\left(x - ik \frac{\sinh \omega t}{\omega}\right) dk.$$

Now, expanding the exponential in $\Omega_0(x)$, regrouping terms and undoing one more Fourier transform one obtains

$$\Omega_0(x)^{-1} f(C^+)\, \Omega_0(x) = \int f(y) \{\exp -(y - x e^{\omega t})^2 / 2\sigma(t)\}\, dy/(2\pi\sigma(t))^{1/2} \tag{3.1}$$

with

$$\sigma(t) = (1 - e^{-2\omega t})/2\omega.$$

Some obvious comments are due. First the semigroup e^{tG} is related to the semigroup e^{tG_0}, $G_0 = \frac{1}{2}(D^2 - \omega^2 x^2)$, by means of the obvious subordination $e^{tG} = e^{+t\omega/2} e^{tG_0}$. The role of the subordination is to allow us to have Ω_0 as a vacuum. Certainly the transition density $p_t^0(x, y)$ of e^{tG_0} is $e^{-t\omega/2} p_t(x, y)$. Also the $p_t(x, y)$ is related to the density $q_t(x, y)$ of Q_t as mentioned above. We only add that the generator $\tilde{G}$ of Q_t is $\frac{1}{2}D^2 - \omega x D$, the generator of the Ornstein–Uhlenbeck process. Thus the oscillator process and the O–U process are related by the canonical transformation generated by $F(x, P) = xP + x^2\omega/2 + t/2$.

(b) *A particle in a constant force field.* The Hamiltonian $H = \frac{1}{2} \sum P_i^2/m_i + E \cdot x$ can be transformed into $H = \frac{1}{2} P_1^2 + a x_1 + \sum_2^n P_1^2/2$ by means of a canonical transformation, thus it suffices to consider $H = \frac{1}{2} P^2 + ax$ together with $G = \frac{1}{2}(d^2/dx^2) + ax$.

Observe now that the generating function

$$F(x, P) = xP + at^2P/2 - axt - t^3a^2/6$$

transforms (x, p) into $(Q, P) = (x + at^2/2, p + at)$ and H into

$$\tilde{H} = H + \frac{\partial F}{\partial t} = \frac{1}{2} P^2.$$

For the free particle,

$$(e^{t\tilde{H}} f)(Q) = \int f(Q')\, e \frac{-(Q'-Q)^2/2t}{(2\pi t)^{1/2}}\, dQ'$$

and therefore, according to the results of Section 2,

$$(e^{tH} f)(x) = e^{atx + a^2t^3/6} (e^{t\tilde{H}} f)(x + at^2/2)$$

which yields for $p_t(x, y)$ the result

$$P_t(x, y) = (2\pi t)^{-1/2} \exp\left\{ -\frac{(x-y)^2}{2t} + \frac{(x+y)\, at}{2} + \frac{a^2t^3}{24} \right\}.$$

(c) *A repulsive oscillator in a constant electromagnetic field.* The Hamiltonian function is given now by

$$H = \frac{1}{2} (P^2 - A(x))^2 + E \cdot x - \frac{\gamma^2}{2} x^2 \tag{3.2}$$

where $A(x) = \Lambda x$, E being a fixed vector and Λ the matrix

$$\Lambda = \omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is such that for any vector N, $\Lambda N = \frac{1}{2} B \wedge N$ with $B = \operatorname{curl} A(x)$ and $\wedge$ denoting the standard vector product in $\mathbb{R}^3$. This example may be transformed into example (a) by means of two canonical transformations. Put $R(t) = \exp t\Lambda$, then with the aid of

$$F(x, \pi) = (\pi, R(t)\, x) = \pi \cdot (R(t)\, x)$$

transform (3.2) into

$$H' = \tfrac{1}{2}(\pi^2 - (q, \sigma^2 q)) + E(t) \cdot q$$

where $E(t) = R(t)\, E$ and σ^2 being a diagonal matrix with diagonal entries $\sigma_1^2 = \sigma_2^2 = \omega^2 + \gamma^2$, $\sigma_3^2 = \gamma^2$. Certainly $q = R(t)\, x$ and $\pi = R(t)\, p$. Now, choose homogeneous but time-dependent vector fields $\xi(t)$ and $\eta(t)$ and a function $\phi(t)$ such that

$$F'(q, P) = q \cdot P + P \cdot \xi(t) - q \cdot \eta(t) + \phi(t)$$

generates the transformation $(Q, P) = (q + \xi(t), \pi + \eta(t))$ and

$$\tilde{H} = H' + \frac{\partial F'}{\gamma t} = \frac{1}{2}(P^2 - (Q, \sigma Q)).$$

It is easy to see that $\dot{\xi} = \eta$, $\dot{\eta} = \sigma^2 \xi + E(t)$ and $\dot{\phi} + \frac{1}{2}(\dot{\xi}^2 + \sigma^2 \xi^2) = 0$, and one can take zero initial conditions all over when integrating these equations.

From example (a) it follows that in the Q coordinates

$$(e^{t\tilde{H}} f)(Q) = \int f(Q')\, \tilde{p}_t(Q, Q')\, dQ'$$

with

$$\tilde{P}_t(Q, Q') = N(t) \exp - \sum \sigma_i \{(Q_i^2 + Q_i'^2) \cosh \sigma_i t - 2 Q_i Q_i'\} / \sinh \sigma_i t$$

and

$$\begin{aligned} N(t) &= (2\pi n_1 n_2 n_3)^{-1/2} \exp - \sum t\sigma_i/2 \\ &= (2\pi \sigma_1 \sigma_2 \sigma_3 / \sinh \sigma_1 \sinh \sigma_2 \sinh \sigma_3\, t)^{1/2}. \end{aligned}$$

Now, taking into account that at $t = 0$ all the canonical transformations considered in this example reduce to the identity, we obtain, undoing all the transformations above, that

$$\begin{aligned} P_t(x, y) = N(t) \exp \Big[& x, R(-t)\, \eta(t) - \phi(t) - \tfrac{1}{2} \sum \{(y_i^2 + (\bar{x}_1(t) + \xi_i)^2) \\ & \times \cosh \sigma_i t - 2 y_i \bar{x}_i\}\, \sigma_i / 2 \sinh \sigma_i t \Big] \end{aligned}$$

where $\bar{x}_i = (R(-t)\,x)_i$. Actually, since the H' above is time dependent, a trivial correction is needed. When obtaining $P'_t(q, q')$ from $\tilde{P}_t(Q, Q')$ one should apply $\tilde{P}_{t-s}\,(Q, Q')$ to the date $\tilde{f}$ obtained from f' by means of (the inverse of) $\tilde{F}(q, P, s)$. This would yield $P'_{s,t}(q, q')$ correctly.

All there examples yield the quantum mechanical expressions when the change $t \to$ it is made, see [5].

(d) We shall mention that there is another class of problems that reduces to the oscillator process, namely, problems leading the forced and damped oscillators characterized by the Newton $\ddot{x} + \lambda\dot{x} + \omega^2 x = f(t)$. These, as well as applications to some infinite-dimensional systems, will appear elsewhere.

(e) Consider now the system with Hamiltonian $H = \frac{1}{2}\sum_i (\sum_j a_{ij}(x)\,p_j)^2$ where the a_{ij} are the components of the Jacobian of ϕ^{-1}: $\mathbb{R}^n \to \mathbb{R}^n$. Consider the canonical transformation

$$F(x, P) = \sum \phi_i(x)\, P_i$$

then $(Q_i, P_i) = (\phi_i(x), \sum_j Q_{ij}(x)\, P_j)$ and $H = \frac{1}{2}\sum P_i^2$. Given $f(x)$,

$$\tilde{f}(Q) = f(\phi^{-1}(Q))$$

and therefore

$$(e^{tH}f)(x) = (e^{t\tilde{H}}\tilde{f})(\phi(x)) = \int f(y)\{\exp - (\phi(y) - \phi(x))^2/2t\}\; J(y)\; dy/(2\pi t)^{n/2}$$

where $J(y) = \det(\partial Q_i/\partial y_j)$.

(f) Consider now a particle diffusing on the unit circle. The corresponding mechanical system has Hamiltonian $H = (I/2)\,L^2$ and $G = (I/2)(\partial^2/\partial\theta^2)$.

In this case $C^+ = \alpha - tI(\partial/\partial\theta)$ and

$$(e^{tH}f)(\alpha) = f(C)\,1 = \sum_n \hat{f}(n)\, e^{-inC^+/2\pi}1 = \sum_n \hat{f}(n)\, e^{-in\alpha/2\pi} e^{-\left(\frac{n}{2\pi}\right)^2 \frac{tI}{2}}$$

where

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, e^{in\alpha}\, dc.$$

A word about these examples: what they have in common is that the canonical transformation reducing the initial problem to a "known" problem induces a transformation of the "spatial coordinates" alone. We hope to complete the treatment of more complicated cases of integrable systems using the results at the end of Section 2 in the near future.

Also, the analogue of these techniques in the context of quantum mechanics will be the subject of a forthcoming note, for it requires a phrasing of its own.

4. Some Moment Theory

In this section we explore some partial aspects of a possible extension of the results in [4]. It is here where the reason for introducing the canonical variables $Q=x$, $P=p+\nabla \ln \Omega_0$ becomes apparent, namely, $P\Omega_0(x)=0$ and $\tilde{\Omega}_0 \equiv 1$ is a vacuum for $\tilde{G}$.

Throughout this section we shall assume that $H=\frac{1}{2}p^2+a(x)\cdot p+V(x)$ and $\tilde{H}=\frac{1}{2}p^2+h(x)\cdot p$, with $h(x)=a(x)-\nabla \ln \Omega_0(x)$. We shall assume also that the equations of motion

$$\dot{Q}_i=\partial\tilde{H}/\partial P_i, \qquad \dot{P}_i=-\partial\tilde{H}/\partial Q_i$$

have a global solution passing through every point (Q, P), and as above, we denote $Q(t, Q, P)$, with P_i replaced by $\partial/\partial Q_i$, by $\tilde{C}^t$.

Let as define the "momentum operators" by

$$P_i(t)=\frac{dC_i^+}{dt}-a_i(C^+), \qquad \tilde{P}_i(t)=\frac{d\tilde{C}^+i}{dt}-h_i(\tilde{C}^+) \tag{4.1}$$

as suggested by the classical relations $\dot{x}_i=p_i+a_i(x)$ and $\dot{Q}_i=P_i+h_i(Q)$.

It is easy to verify that

$$P_i(t)\,\Omega_0=\frac{\partial \ln}{\partial x_i}\Omega_0(C^+)\,\Omega_0, \qquad \tilde{P}_i(t)\;=\frac{\partial}{\partial Q_i}1=0. \tag{4.2}$$

It suffices to differentiate $Q_t x_i$ and to look at it. Similar computations are contained in

Lemma 4.3. *The following hold*

(i) $[f(C^+), g(C^+)]\,\Omega_0=0, \qquad [f(\tilde{C}^+), g(\tilde{C}^+)]\,1=0$

for appropriate but arbitrary f, g.

(ii) $[C_i^+, p_j]\,\Omega_0=-\delta_{ij}\Omega_0, \qquad [\tilde{C}_i, P_j]\,1=-\delta_{ij}1$

(iii) $[p_i, p_j]\,\Omega_0=0, \qquad [P_i P_j]\,1=0.$

Proof.

(i) $\Omega_0^{-1}(f(C^+)\,g(C^+)\,\Omega_0)=\Omega_0^{-1}P_t fg\Omega_0=\Omega_0^{-1}(g(C^+)f(C^+)\,\Omega_0)$

(ii) $$\Omega_0^{-1}C_i^+\,P_j\,\Omega_0=\Omega_0^{-1}C_i^+\,\frac{\partial \ln}{\partial x_j}\,\Omega_0(C^+)\,\Omega_0=\Omega_0^{-1}\,\frac{\partial \ln}{\partial x_j}\,\Omega_0(C^+)\,C_i^+\,\Omega_0$$

$$\times\,\Omega_0^{-1}\left(\frac{d}{dt}\,C_j^+-a_j(C^+)\right)C_i^+\,\Omega_0=\Omega_0^{-1}\,\frac{d}{dt}\,(C_j^+\,C_i^+)\,\Omega_0$$

$$-\,\Omega_0^{-1}C_j^+\,\frac{d}{dt}\,C_i^+\,\Omega_0-\Omega_0^{-1}a_j(C^+)\,C_i^+\,\Omega_0$$

but from

$$\frac{d}{dt}\,C_i\Omega_0=\frac{\partial \ln}{\partial x_i}\,\Omega_0(C^+)\,\Omega_0+a_i(C^+)\,\Omega_0$$

and

$$\Omega_0^{-1}\,\frac{d}{dt}\,(C_j^+\,C_i^+)=\frac{\partial}{\partial t}\,Q_t x_i x_j=A_t\,\tilde{G}x_i x_j$$

$$=\delta_{ij}+\Omega_0^{-1}\left\{C_0^{-1}\,\frac{\partial \ln}{\partial x_j}\,\Omega_0(C^+)+C_i^+\,a_j(C^+)+C_j^+\,a_i(C^+)\right.$$

$$\left.+\;C_j^+\,\frac{\partial \ln}{\partial x_i}\,\Omega_0(C^+)\right\}\Omega_0$$

it follows that

$$\Omega_0^{-1}P_jC_i^+\,\Omega_0=\delta_{ij}+\Omega_0^{-1}\,C_i\,\frac{\partial \ln}{\partial x_j}\,\Omega_0(C^+)\,\Omega_0$$

and therefore

$$\Omega_0^{-1}\{C_i^+\,P_j\Omega_0+P_jC_i^+\,\Omega_0\}=\Omega_0^{-1}[C_i^+\,,P_j]\,\Omega_0=-\delta_{ij}.$$

The rest are left for the reader. ▮

Now put

$$C_i(t)=p_i(t)-\frac{\partial \ln}{\partial x_i}\,\Omega_0(C^+),\qquad \tilde{G}_i(t)=\tilde{P}_i(t) \tag{4.4}$$

then Lemma 4.3 implies that

$$\begin{gathered}[C_i^+\,,C_j^+\,]\,\Omega_0=[C_i,\,C_j]\,\Omega_0=[\tilde{C}_i^+\,,\tilde{C}_j^+\,]\;1=[\tilde{C}_i,\,\tilde{C}_j]\;1=0\\ [C_i,\,C_j^+\,]\,\Omega_0=\delta_{ij}\Omega_0,\qquad [\tilde{C}_i(t),\,\tilde{C}^+(t)]\;1=\delta_{ij}1\end{gathered} \tag{4.5}$$

for $i, j = 1, \ldots, n$. And we also have

$$C_i(t)\,\Omega_0 = \tilde{C}_i(t)\,1 = 0 \tag{4.6}$$

for $i = 1, \ldots, n$. Now it is apparent that the notation was chosen to conform to that of quantum mechanics.

Define now, for any multi-index $(m) = (m_1, \ldots, m_n)$,

$$\begin{aligned} h_0 &= \Omega_0, & \tilde{h}_0 &= 1 \\ h_m(x,t) &= C^+(t)^{(m)}\,\Omega_0, & \tilde{h}_{(m)}(x,t) &= (C^+(t))\,1 \end{aligned} \tag{4.7}$$

then from (4.6) and (4.5) we obtain an analogue to Proposition 10 in [4].

PROPOSITION 4.8.

$$\begin{array}{lll} \text{(a)} & C_i^+ h_{(m)} = h_{(m+e_i)}, & \tilde{C}_i^+ \tilde{h}_{(m)} = \tilde{h}_{(m+e_i)} \\ \text{(b)} & C_i h_{(m)} = m_i h_{(m-e_i)}, & \tilde{C}_i \tilde{h}_{(m)} = m_i \tilde{h}_{(n-e_i)} \\ \text{(c)} & \dfrac{\partial h}{\partial t}(m) = G h_{(m)}, & \dfrac{\partial \tilde{h}}{\partial t}(m) = \tilde{G}\tilde{h}_{(m)} \\ \text{(d)} & C_i^+ C_i h_{(m)} = m_i h_{(m)}, & \tilde{C}_i^+ C_i h_{(m)} = m_i \tilde{h}_{(m)}. \end{array}$$

Proof. Let us do (part of) (c):

$$\begin{aligned} \frac{\partial h}{\partial t}(m) &= \Omega_0 \frac{\partial}{\partial t}\Omega_0^{-1}(C^+)^{(m)}\,\Omega_0 = \Omega_0 \frac{\partial}{\partial t} Q_t x^{(m)} = \Omega_0 \Omega_0^{-1} \tilde{G} P_t x^{((m)}\Omega_0 \\ &= G\Omega_0(\Omega_0^{-1}(C^+)^{(m)})\,\Omega_0 = G h_{(m)}. \quad \blacksquare \end{aligned}$$

Comment. Proposition 2.20 contains the relationship between $h(x(t))$ and $\tilde{h}(x, t)$. We mention in passing that, by starting with a vacuum $\tilde{\Omega}_0$ for $\tilde{G}$, one can obtain still more moment systems, and this is a good point to stop at for the time being.

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